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ORBITAL OPERATIONS STUDIES

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Natural Resources Research Institute
in association with the
National Aeronautics and Space Administration

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ENGINEERING Education - Research - Service
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ORBITAL OPERATIONS STUDY
Technical Report No. 3

THE CHARACTER OF GRAVISPHERES

Thomas C. Ensign
John C. Bellamy
October 1965

This is the third of a series of technical reports of an Orbital Operations Study by the Natural Resources Research Institute sponsored by Grant No. NsG 658 of the National Aeronautics and Space Administration. The overall purpose of this study is to help establish more productively effective ways of controlling orbital operations; the work reported upon here is meant to help define the characteristics of those regions in which orbital operations are conducted.

This report is to be submitted for publication elsewhere in order to present these novel concepts and nonmenclature for the critical attention of a wide audience. In the meantime copies of this mimeographed preprint are available upon demand from the NRRI.

Discussions with Dr. Willis L. Everett and Mr. Roland Lamberson have contributed significantly to the preparation of this report, and Mr. Larry A. Baccari and Mr. Robert O. Lamb assisted in checking the numerical computations and drafting the illustrations.

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THE CHARACTER OF GRAVISPHERES

BRIEF

LIST OF ILLUSTRATIONS

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REFERENCES

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2. "Report to the Executive Committee of the Working Group on the System of Astronomical Constants" International Astronomical Union, 28 February 1964.

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THE CHARACTER OF GRAVISPHERES

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1. Purpose

The work reported upon here is part of an Orbital Operations Study, the purpose of which is to help establish more productively effective ways of controlling orbital operations. It consists largely of investigations of more effective ways of portraying and utilizing the large amounts of data which are characteristic of orbital operations. Much of this work is thus concerned with the interrelationships among the many aspects of orbital and related operations and hence is concerned with useful functional definitions of the major kinds of operations involved. It has been recognized that a conveniently useful way of doing so is to identify them with characteristic operational regions, and the specific purpose here is to identify and define the general character of those regions in which orbital operations are conducted.

For example, the operations of the Army, Navy and Air Force and the sciences of geology, oceanograph and meteorology are organized and identified with respect to the lithosphere, hydrosphere and atmosphere. By analogy, the names "pyrosphere" and "gravisphere" have been suggested[1] for identifying the characteristic regions of "space" or "orbital" operations and "space science" in accordance with the following definition.

Operational Regions

The "earth, water, air and fire dependent" Biospheres or
the "full of solid, liquid, gas or radiation"
Lithospheres, Hydrospheres, Atmospheres or Pyrospheres
of the regions of dominant gravitational influence, or Gravispheres,
of the Moons, Planets, Stars or Galaxies.

SPHERES OF INTEREST

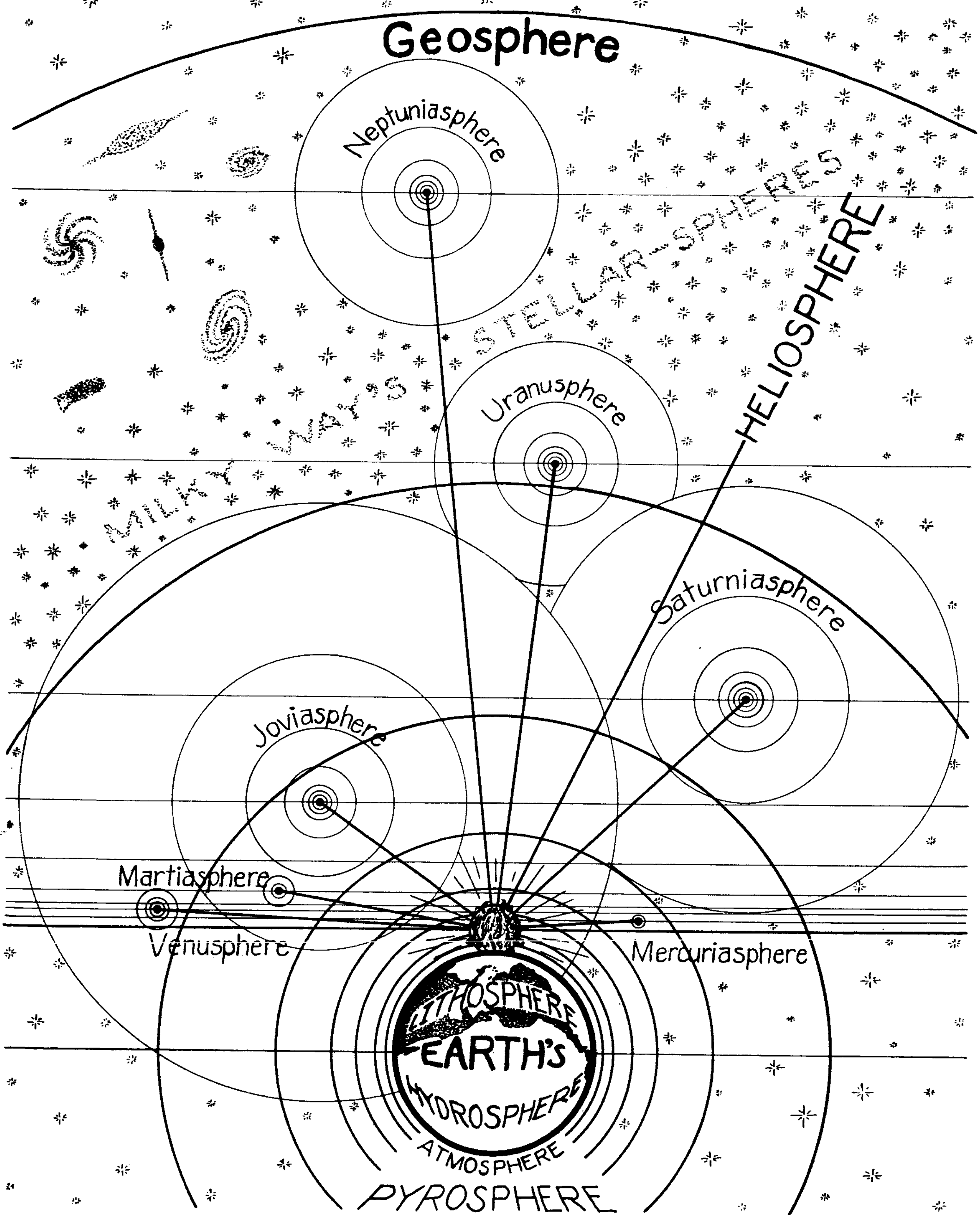


FIGURE I

The qualitative concepts involved in this definition are illustrated in Figure 1. This drawing is intended to reflect a concept of the "sea of space" as consisting of large galactic eddies in which smaller stellar-system eddies are embedded and sweep with them even smaller planetary eddies in which moon-sized eddies are in turn embedded. The extent of each such eddy is considered to have been made visible with swarms of artificial satellites in nearly circular orbits around the hard-core islands of stars, planets or moons at their centers. The circles in the figure are thus indicative of possible orbits around the planets, and the horizontal lines are indicative of the orbits of the planets and their satellites around the Sun. The outermost circles are indicative of the outermost of possible orbits around the planets, or hence are indicative of the extent of their "gravispheres", as they would be seen in perspective proportion by an observer on the Moon

Names such as the "Geosphere, Lunasphere or Heliosphere" and phrases such as "the gravisphere of the Earth, Moon or Sun" are suggested here as being usefully synonymous ways of identifying the "Earth-, Moon- or Sun-centered region of the universe". These identifications reflect the fact that gravitational influences are dominant to greater distances from the centers of such bodies than any of their other known physical properties. In addition, the extent and character of their gravitational fields are of major operational and scientific importance now that artificial satellites are being placed in permanent, gravitationally determined, orbits around them for both applied and purely scientific purposes.

Each such "gravisphere" or "body-sphere" can then usefully be subdivided into its lithosphere, hydrosphere, atmosphere and whatever its remaining "empty space" might be called. The name "pyrosphere" is suggested for that outermost portion of gravispheres since, rather than being "empty", it is characteristically full of the radiation from the "fires" of the Sun and stars. Indeed, it is full of such "fiery" radiation by virtue of the fact that it is essentially empty of the other radiation-absorbing solid, liquid and gaseous states of the matter-energy substance. The use of the prefix "pyro" also consistently completes the identification of the four basic sub-regions of gravispheres with the Greek, "litho, hydro, atmo and pyro" identifications of the four basic elements of life.

The characteristic regions of orbital operations can thus well be identified as being the "pyrospheres" of particular "gravispheres" or "body-spheres". For example, the "full-of-fiery-radiation" connotation of "pyrosphere" directly implies an absence of significant amounts of matter and hence the possibility of permanent satellite orbits. It is also directly connotative of both the major source of operational difficulty and the major subject of scientific interest within this "fiery" region.

The purpose here is thus to define the extent and character of gravispheres and pyrospheres in more quantitative detail. Specifically, the purpose here is to compute and portray the distributions of the field of gravity around celestial bodies and, thereby, to determine the extent of the common outer limit of their pyrospheres and gravispheres. The companion problem of establishing a quantitative definition of the location of the inner boundary between the airless pyrospheric regions of orbital operations and the atmospheric regions of air operations is deferred for later work.

2. Orders of Magnitude

It is informative at the outset to illustrate the degree to which the gravitational field is dominated by the mass of the Earth in its near vicinity. In accordance with Newton's Law of Gravitation, the attractive force, F_e, between a mass m at a distance R from the center of the mass of the Earth, M_e, is expressed in terms of the universal gravitational constant, G, by the equation

$$F_e = GmM_e/R^2 \tag{1}$$

The same mass m at a distance r from the center of the mass of the Sun, M_s, would similarly be attracted toward the Sun with a force F_s given by

$$F_s = GmM_s/r^2 \tag{2}$$

The relative magnitude of these two forces is then indicated by the ratio

$$F_e/F_s = (M_e/M_s)(r/R)^2 \tag{3}$$

This ratio can be evaluated with the astronomical constants,[2]

$$M_s/M_e = 332,958$$

and, for the equatorial radius, $R = R_e$, of the Earth, measured in terms of the astronomical unit A, by

$$\begin{aligned} R_e/A &= 8.794 \text{ seconds of arc of solar parallax} \\ &= 4.263 \times 10^{-5} \text{ radians} \\ &= 1/23,460 \end{aligned}$$

Substitution of these constants in Equation 3 indicates that the Earth's Newtonian attraction is about 1,700 times stronger than that of the Sun on the surface of the Earth.

At first sight it might then be considered that the dominance of the Earth's gravitational field would extend out to that radius, R, at which the forces of Equations 1 and 2 toward the Earth and Sun become numerically equal, or out to where the value of R satisfies the equality

$$M_e/R^2 = M_s/(A - R)^2 \tag{4}$$

Since the distance R is much smaller than the distance A, these two forces become equal at about where

$$R = A\sqrt{M_e/M_s} \tag{5}$$

or at about 1/580 of the distance, A, from the Earth to the Sun.

On second sight it is thus seen that some other criterion must be used to determine the extent of gravispheres. That is, the orbit of the Moon around the Earth is obviously dominated by the Earth's gravity. On the other hand, the mean distance of the Moon from the Earth of 0.0026 astronomical units[3] is about 1/380 of the distance from the Earth to the Sun. Consequently the distance to the Newtonian neutral point of Equations 4 or 5 is only about 380/580, or less than seven tenths of the distance to the obviously Earth-centered orbit of the Moon around the Earth.

3. Two-Body Coordinate System

As indicated by the concepts illustrated in Figure 1, however, the Earth's gravisphere can better be considered to be a relatively small "eddy" of possible orbits around the Earth, all of which are embedded in and swept along in the larger eddy of possible planet-like orbits around the Sun. Consequently the orbital effect of the Earth's gravity is evidently to be compared more with the spatial variations of the Sun's effects than with their absolute magnitude. The effect of the absolute magnitude of the Sun's gravity is primarily "to sweep" all masses in the vicinity of the Earth around the Sun in virtually the same planet-like orbit.

A convenient and straightforward way of dealing with this problem is thus to utilize a two-body coordinate system such as illustrated in Figure 2. This coordinate system is defined to be attached to both the larger "central" body of mass M_c such as the Sun and the smaller "revolving" body of mass M such as the Earth. It thus rotates with that vectorial angular velocity $\overleftarrow{\omega}$ with which two such celestial bodies revolve around their common center of mass at a mean distance a between them in accordance with Kepler's Law that

$$\omega^2 a^3 = G(M_c + M) \tag{6}$$

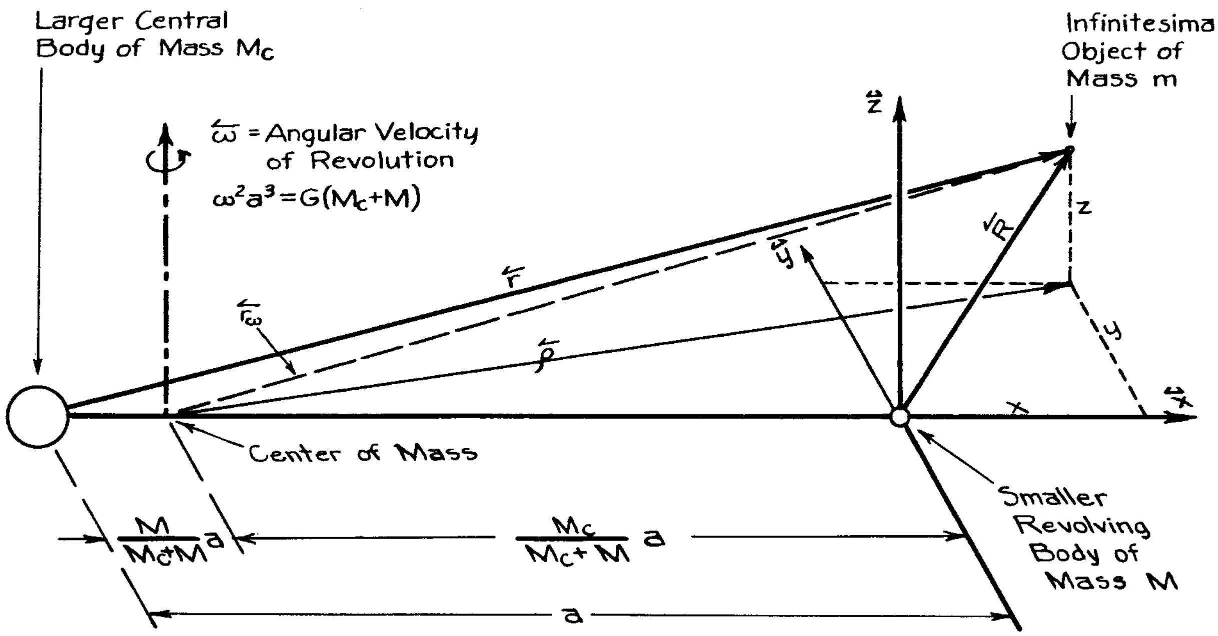
In such a rotating coordinate system, the vectorial equation of motion for an infinitesimal mass m such as of an artificial satellite can be written as

$$d\overleftarrow{v}/dt = \overleftarrow{g} + \overleftarrow{g}_c - \overleftarrow{\omega}x[\overleftarrow{\omega}x\overleftarrow{r}_\omega] - 2\overleftarrow{\omega}x\overleftarrow{v} + \overleftarrow{F}/m \tag{7}$$

In this equation;

$d\overleftarrow{v}/dt$ stands for the acceleration, or force per unit mass, of the mass m evaluated as the time rate of change of its velocity $\overleftarrow{v}$ with respect to this coordinate system;

TWO-BODY COORDINATE SYSTEM



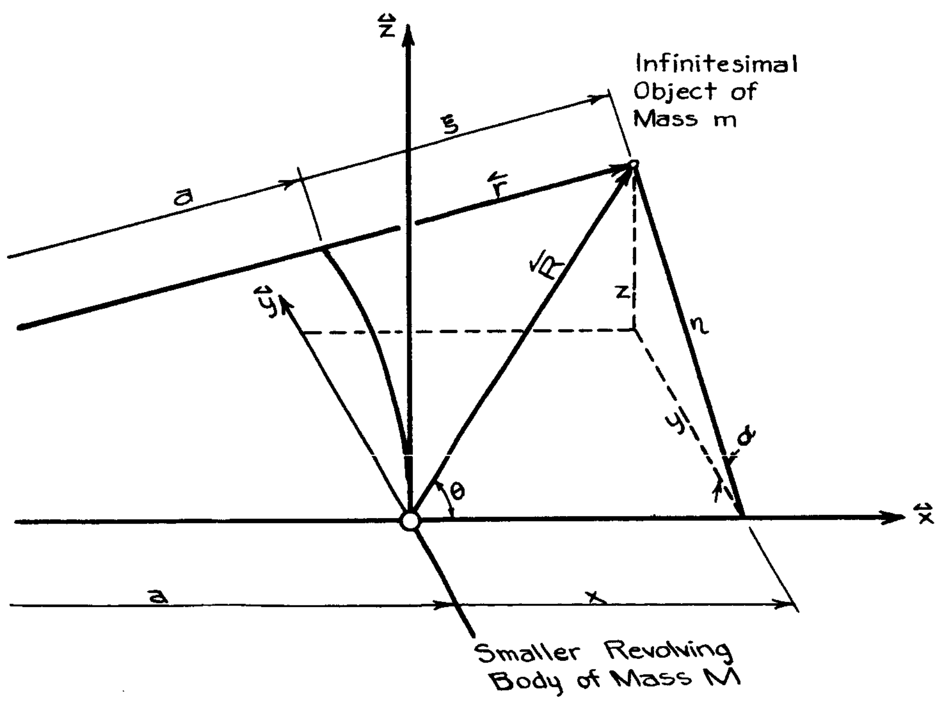


FIGURE 2

$$\vec{g} = -(GM/R^2)\hat{R} \tag{8}$$

stands for the Newtonian tendency for acceleration, or force per unit mass, acting on an infinitesimal mass m at a distance R and in the negative of its direction (or of the unit vector $\hat{R}$) from the smaller body of mass M;

$$\vec{g}_c = -(GM_c/r^2)\hat{r} \tag{9}$$

likewise stands for the Newtonian acceleration of m at a distance r and in the negative of the $\hat{r}$ direction from the larger body of mass M_c;

$$-\vec{\omega}x[\vec{\omega}x\vec{r}_\omega] = \omega^2\vec{\rho} \tag{10}$$

stands for the centrifugal force per unit mass acting on a mass m at a vectorial distance $\vec{r}_\omega$ from the center of rotation of the coordinate system, or at the perpendicular vectorial distance, $\vec{\rho} = \rho\hat{\rho}$, from its axis of rotation;

$-2\vec{\omega}x\vec{v}$ stands for the Coriolis force per unit mass acting on a mass m with a velocity $\vec{v}$ with respect to such a rotating coordinate system; and

$\vec{F}/m$ stands for any other force per unit mass (such as electromegnetic forces, propulsion forces, drag forces, perturbing gravitational forces of other masses, etc.) which might be acting upon the infinitesimal mass m of interest.

The following form of the equation of motion of a mass m in this two-body coordinate system is convenient for subsequent considerations, and has been obtained by rearranging and substituting for the terms in Equation 7 in accordance with Equations 6, 8, 9 and 10.

$$\frac{d\vec{v}}{dt} + 2\vec{\omega}x\vec{v} - \frac{\vec{F}}{m} = \vec{f} = -\frac{GM}{R^2}\hat{R} - \frac{GM_c}{r^2}\hat{r} + \frac{G(M_c + M)}{a^3}\vec{\rho} \tag{11}$$

4. Neutral Points

A start toward defining the extent of the gravisphere of the smaller body M can then be obtained by evaluating the distance R from its center to the two "neutral points" on the x-axis of Figure 2 at which both sides of Equation 11 become zero. These are "neutral points" in the sense that it is there that the sum of the Newtonian attractions of the two bodies and the centrifugal force on the right side of Equation 11 are balanced.

Considering first the external neutral point on the extension of the line through the two bodies of interest, it is seen from Figure 2 that, external to M,

$$\rho = \frac{M_c}{M_c + M} a + R, \qquad r = a + R$$

and the unit vectors $\hat{r}$, $\hat{\rho}$ and $\hat{R}$ are all in the same direction. Consequently, by substituting for ρ and r in the right side of Equation 11, the external neutral point occurs at that distance R for which

$$\frac{M}{R^2} + \frac{M_c}{(a + R)^2} - \frac{M + M_c}{a^3}\left[\frac{M_c}{M + M_c} a + R\right] = 0$$

or where the ratio of distances, R/a, satisfies the equations

$$\frac{M}{(R/a)^2} + \frac{M_c}{(1 + R/a)^2} - \left[M_c + (M + M_c)(R/a)\right] = 0$$

or $$(R/a)^3 = \frac{M}{3M_c} \frac{(1 + R/a)^2(1 - R^3/a^3)}{(1 + R/a + R^2/3a^2)} \tag{12}$$

Similarly for the internal neutral point on the line between the two bodies of interest,

$$\rho = \frac{M_c}{M_c + M} a - R, \qquad r = a - R$$

and the unit vectors $\hat{r}$ and $\hat{\rho}$ are in the opposite direction of $\hat{R}$. This internal neutral point thus occurs at that distance, R, for which

$$\frac{M}{R^2} - \frac{M_c}{(a - R)^2} + \frac{M + M_c}{a^3}\left[\frac{M_c}{M_c + M} a - R\right] = 0$$

or where the ratio, R/a, satisfies the equation

$$(R/a)^3 = \frac{M}{3M_c} \frac{(1 - R/a)^2 (1 - R^3/a^3)}{(1 - R/a + R^2/3a^2)} \tag{13}$$

These relationships have been expressed in this form to utilize the fact that the ratio R/a is usually much smaller than unity, and hence that it is convenient to define a "nominal radius, R_G, of neutral points", by the equation

$$R_G = a\sqrt[3]{M/3M_c} \tag{14}$$

For example, for the ratio M_c/M of the masses of the Sun and the Earth[2] of 332,958, the ratio R_G/a is equal to the reciprocal of the cube root of 998,874 or very nearly 1/100. Consequently the Earth's interior and exterior neutral points can both be considered to occur at 1/100 of the distance from the Earth to the Sun within an accuracy of about one third of one percent. In comparison, the Moon is at about 100/380, or slightly more than one quarter, of the distance to these neutral points.

It is noteworthy in this regard that Equations 12, 13 and 14 can be rigorously combined into the equations

$$R^3 = R_G^3 \frac{(1 \pm \epsilon)^2 (1 - \epsilon^3)}{(1 \pm \epsilon + \epsilon^2/3)} \tag{15}$$

where

$$\epsilon = R/a \tag{16}$$

Since ϵ is much less than unity, a first approximation, R', of the distance R to either of the neutral points is then given by $R' = R_G$. For more precise considerations, a second order approximation, R'', for a particular neutral point can then be computed directly and easily by substituting values of R_G/a for ϵ in Equation 15. In those infrequent cases in which even more precise determinations might be desired, a third order approximation, R''', can be computed by substituting values of R''/a for ϵ in Equation 15, and so forth.

5. Normalized Equations of Motion

Apparently, then, this distance to the neutral points of two mutually revolving bodies can be considered to be the extent of the smaller body's gravisphere in the directions of the line between them. In order to determine the extent of this gravisphere in other directions, however, it is necessary to consider the equations of motion in more detail. As demonstrated below, it is conveniently useful to do so by normalizing these equations in terms of either the characteristic distance a between the bodies of interest or the characteristic neutral point distance R_G.

It is convenient for this purpose to combine the terms for the $\hat{r}$ and $\hat{\rho}$ directions on the right side of Equation 11 in accordance with the vector relationship implied by Figure 2 that

$$\vec{\rho} = \vec{r} - \vec{z} - \frac{M}{M_c + M} a\hat{x} \tag{17}$$

or hence that

$$\vec{f} = -\frac{GM}{R^2}\hat{R} + \left[\frac{G(M_c + M)}{a^3} r - \frac{GM_c}{r^2}\right]\hat{r} - \frac{G(M + M_c)}{a^3}\hat{z} - \frac{GM}{a^2}\hat{x} \tag{18}$$

The bracketed term in this equation can then be simplified by expressing it in terms of a distance

$$\xi = r - a \tag{19}$$

measured in the $\hat{r}$ direction from the r = a sphere, as

$$\begin{aligned}
\left[\frac{G(M_c + M)}{a^3} r - \frac{GM_c}{r^2}\right] &= \frac{GM_c}{a^3(a + \xi)^2}\left[\left(1 + \frac{M}{M_c}\right)(a + \xi)^3 - a^3\right] \\
&= \frac{GM_c}{a^3(a + \xi)^2}\left[3a^2\xi + 3a\xi^2 + \xi^3 + \frac{M}{M_c}(a + \xi)^3\right] \\
&= \frac{GM_c}{a^2}(3\xi/a)\frac{(1 + \xi/a + \xi^2/3a^2)}{(1 + \xi/a)^2} + \frac{GM}{a^2}(1 + \xi/a) \\
&= \frac{GM_c}{a^2}(3\xi/a)\left[1 - (\xi/a)\frac{(1 + 2\xi^2/3a^2)}{(1 + \xi/a)^2}\right] + \frac{GM}{a^2}(1 + \xi/a)
\end{aligned}$$

Similar normalization of the other terms of Equation 18 with respect to the distance a then leads to the equation

$$\frac{\overleftarrow{f}}{GM_c/a^2} = -\frac{M}{M_c}\frac{\overleftrightarrow{R}}{(R/a)^2} + 3\left(\frac{\xi}{a}\right)\left[1 - \left(\frac{\xi}{a}\right)\frac{(1 + 2\xi/3a)}{(1 + \xi/a)^2}\right]\overleftrightarrow{r} - \frac{z}{a}\overleftrightarrow{z}$$

$$+ \frac{M}{M_c}\left[(1 + \frac{\xi}{a})\overleftrightarrow{r} - \overleftrightarrow{x} - \frac{z}{a}\overleftrightarrow{z}\right] \tag{20}$$

Since this form of the equation of motion has been normalized with respect to the distance a between the two bodies of interest, it is primarily useful for considering the degree to which the smaller body perturbs the gravitational field of the larger central body. An alternate form which is more indicative of the pertubation by the larger body of the gravitational field in the vicinity of the smaller one can be obtained by normalizing with respect to the neutral point distance

$$R_G = a\sqrt[3]{M/3M_c} \tag{14}$$

and the characteristic ratio

$$K = R_G/a = \sqrt[3]{M/3M_c} \tag{21}$$

The result is that

$$\frac{\overleftarrow{f}}{GM/R_G^{\ 2}} = \frac{-\overleftrightarrow{R}}{(R/R_G)^2} + (\xi/R_G)\overleftrightarrow{r} - \frac{(z/R_G)}{3}\overleftrightarrow{z}$$

$$-K\left(\frac{\xi}{R_G}\right)^2\frac{(1 + 2K\xi/3R_G)}{(1 + K\xi/R_G)^2}\overleftrightarrow{r} + K^2\left[\overleftrightarrow{r} - \overleftrightarrow{x}\right] + K^3\left[\frac{\xi}{R_G}\overleftrightarrow{r} - \frac{z}{R_G}\overleftrightarrow{z}\right] \tag{22}$$

The extent to which the central body M_c perturbs the gravitational force, $-\overleftrightarrow{R}/(R/R_G)^2$, of the smaller body M is indicated largely by the second term in the braces of the following approximate form of Equation 22;

$$\frac{\overleftarrow{f}}{GM/R_G^{\ 2}} \doteq \frac{1}{(R/R_G)^2}\left\{-\overleftrightarrow{R} + \left(\frac{R}{R_G}\right)^3\left[(\xi/R)\overleftrightarrow{r} - (z/R)\frac{\overleftrightarrow{z}}{3}\right]\right\} \tag{23}$$

The factors (ξ/R) and (z/R) correspond to directional cosines whose values range from zero to one; the term in the bracket thus essentially defines the direction of the perturbing force; and the factor $(R/R_G)^3$ approximately represents the magnitude of that perturbing force with respect to a unit of magnitude of the unperturbed gravitational force in the $-\overleftrightarrow{R}$ direction.

For the Earth-Sun system, the relative magnitude of these solar perturbations are of thus seen to be of the order of

- o 1 part in 1 at distances $R = R_G$ from the Earth of 1/100 of the distance to the Sun or of 235 times the radius of the Earth;
- o 1 part in $(380/100)^3$ or about 60 at the distance of the Moon of about 1/380 of the distance to the Sun;
- o 1 part in $(235/6.62)^3$ or about 43,000 at the distance of synchronous or one-day-period orbits; and
- o 1 part in $(235)^3$ or about 13,000,000 at the surface of the Earth.

6. Potential Fields

In order to obtain a clearer picture of the perturbing effects of the central body in the vicinity of planets or moons, it is useful to represent their gravitational forces in terms of the gradient of a single scalar quantity called the gravitational potential.

Specifically the gravitational potential, Φ, is defined here to be that scalar quantity whose vector gradient $\overleftrightarrow{\nabla}\Phi$ is equal to the negative of both sides of Equation 11. Or, referring to the equivalent expression in Equation 22, Φ is defined to be that quantity for which

$$\overleftrightarrow{\nabla}\Phi/(GM/R_G{}^2) = -\overleftrightarrow{f}/(GM/R_G{}^2)$$

$$= \frac{\overleftrightarrow{R}}{(R/R_G)^2} - (\xi/R_G)\overleftrightarrow{r} + \frac{(z/R_G)}{3}\overleftrightarrow{z}$$

$$+ K(\xi/R_G)^2 \frac{(1 + 2K\xi/3R_G)}{(1 + K\xi/R_G)^2}\overleftrightarrow{r} - K^2(\overleftrightarrow{r} - \overleftrightarrow{x}) - K^3\left(\frac{\xi}{R_G}\overleftrightarrow{r} - \frac{z}{R_G}\overleftrightarrow{z}\right) \quad (24)$$

Consequently, as checked by vectorial differentiation,

$$\frac{\Phi}{GM/R_G} = \frac{\Phi_o}{GM/R_G} - \frac{1}{R/R_G} - \frac{(\xi/R_G)^2}{2} + \frac{(z/R_G)^2}{6}$$

$$+ \frac{K}{3}\frac{(\xi/R_G)^3}{1 + K\xi/R_G} - K^2\left(\frac{\xi}{R_G} - \frac{x}{R_G}\right) - \frac{K^3}{2}\left[(\xi/R_G)^2 - (z/R_G)^2\right] \quad (25)$$

where Φ_o is an arbitrarily assignable constant of integration.

The distribution of this gravitational potential in the ecliptic ($z = 0$), axial ($y = 0$) and tangential ($x = 0$) planes has been mapped in Figures 3, 4 and 5, respectively, and has been summarized in the isometric view of Figure 6. The constant potential lines of these figures were obtained by adding numerical values of the major terms of Φ in accordance with the equations

$$\Phi = \Phi_R + \Phi_\xi + \Phi_z + \Phi_o$$

where $\frac{\Phi_R}{\Phi_1} = -\frac{1}{R/R_G}$; $\frac{\Phi_\xi}{\Phi_1} = -\frac{(\xi/R_G)^2}{2}$ $\frac{\Phi_z}{\Phi_1} = \frac{(z/R_G)^2}{6}$

$$\Phi_o = \frac{3}{2}\Phi_1 \text{ and } \Phi_1 = GM/R_G \tag{26}$$

This addition was accomplished with the intersection technique of graphically adding circular plots of selected constant values of Φ_R/Φ_1 and straight line plots of selected constant values of Φ_ξ/Φ_1 and Φ_z/Φ_1. The use of straight line plots for constant values of Φ_ξ/Φ_1 is consistent with the approximation implied by Equation 26 that values of K in Equation 25 are negligibly small with respect to unity.

These figures, by virtue of the way in which they have been normalized, thus portray the gravitational field in the vicinity of the smaller of <u>any</u> two bodies for which the characteristic ratio $K = \sqrt[3]{M/3M_c}$ is negligibly small with respect to unity. The zero-reference level of potential has been assigned as being the potential at the nominal neutral points (where $R = |\xi| = R_G$ and $z = 0$) by assigning the value of 3/2 to the normalized constant of integration, Φ_o/Φ_1.

The effect of neglecting the value of K with respect to unity in Figure 3 is illustrated by the plot in Figure 7 of the gravitational field in the Moon's ecliptic plane. This is an extreme example since, as indicated in Table II of Section 8, the value of K = 0.16 for the Earth-Moon system is much larger than for any of the Sun-Planet systems. The solid lines in Figure 6 were drawn for the value of $\Phi_\xi/\Phi_1 = -(\xi/R_G)^2/2$ specified by Equation 26 without, however, also approximating Earth-centered circles of constant ξ by straight lines as is the case in Figure 3. The effect of neglecting the higher order terms in K of Equation 25 is illustrated by including the first order K term in the values of

$$\Phi_\xi/\Phi_1 = -\frac{(\xi/R_G)^2}{2} + \frac{K}{3}(\xi/R_G)^3$$

used to draw the dashed lines of Figure 7.

FIELD OF GRAVITATIONAL POTENTIAL
ECLIPTIC PLANE

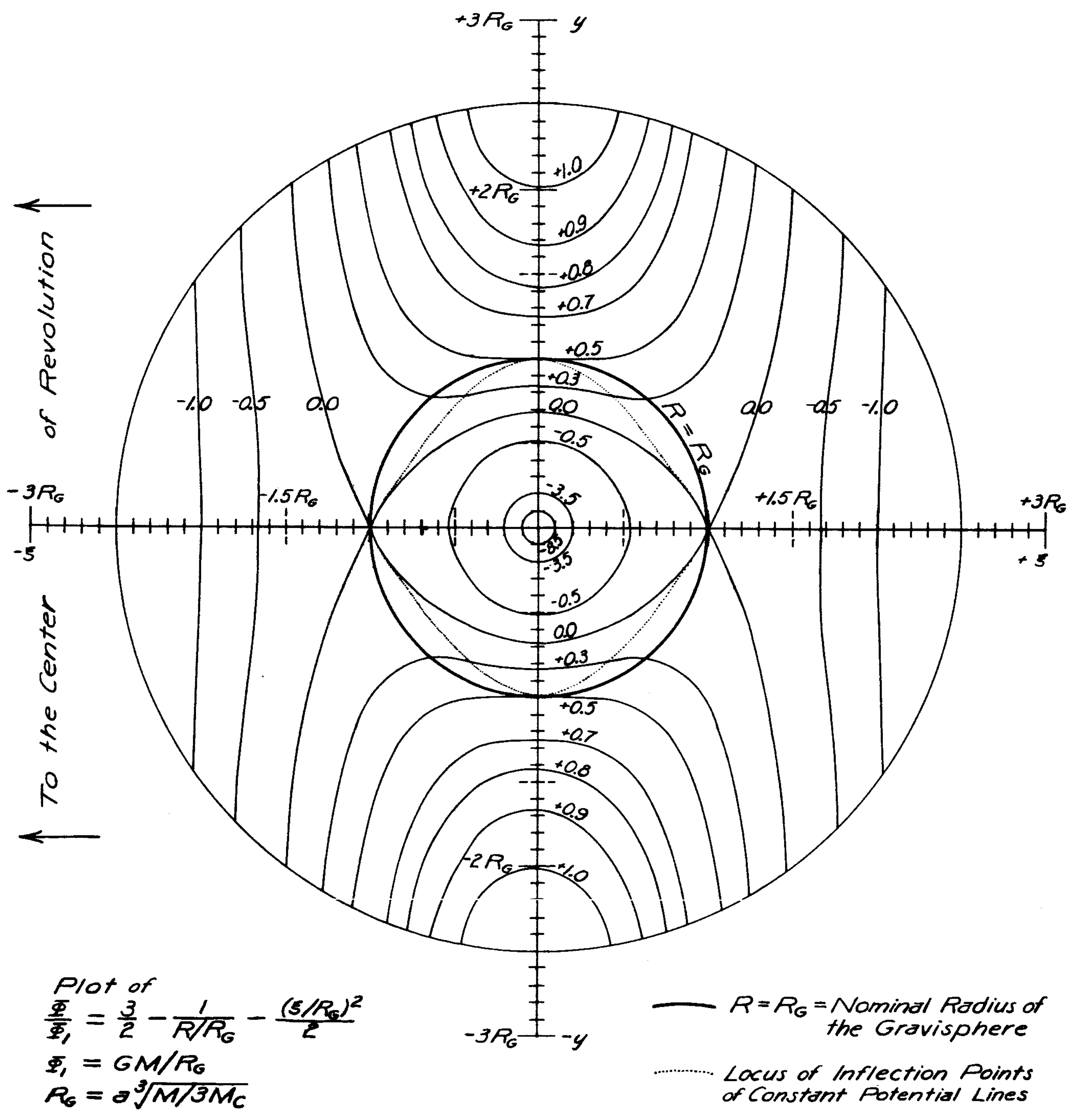
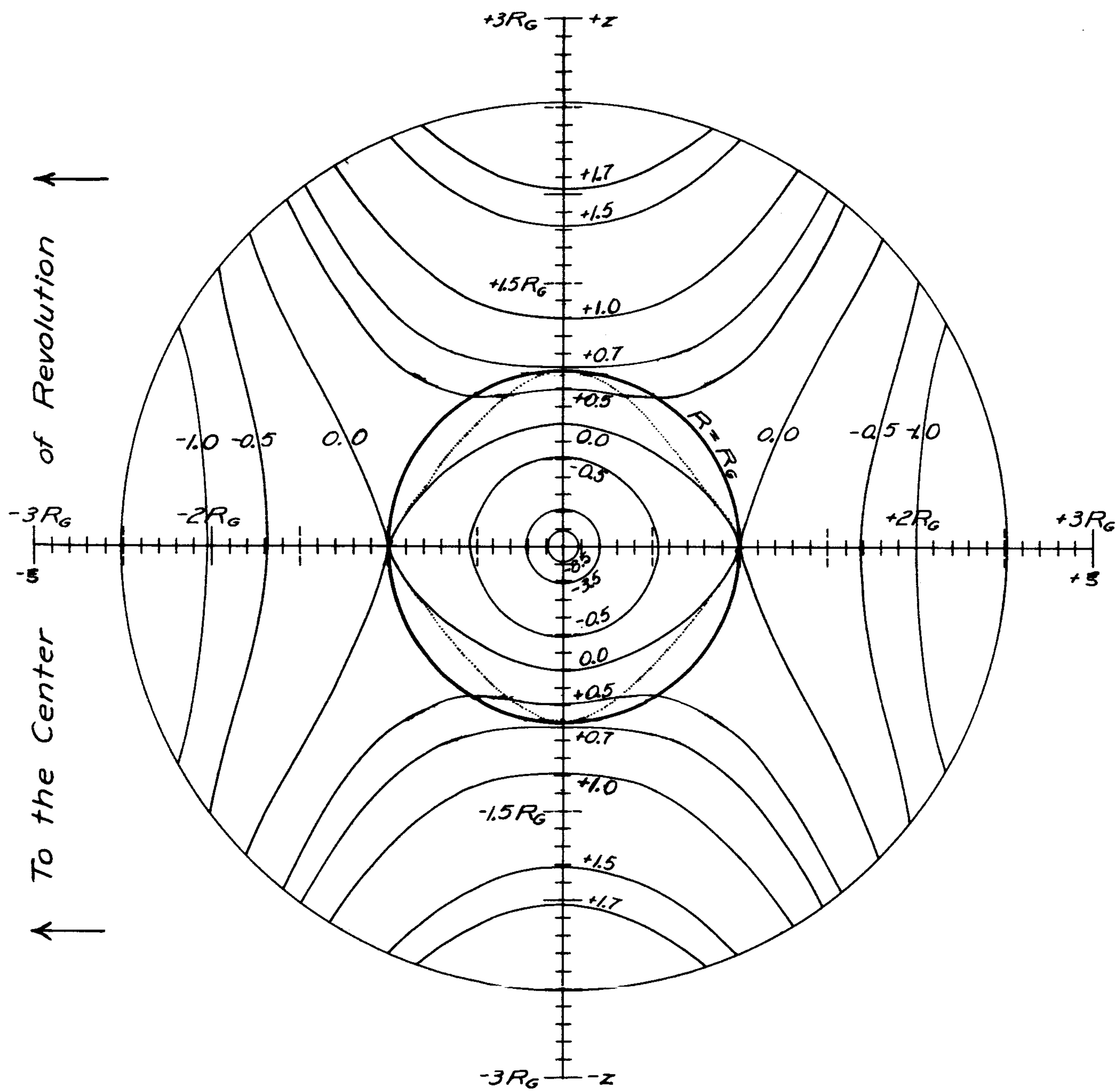


FIGURE 3

FIELD OF GRAVITATIONAL POTENTIAL
AXIAL PLANE



Plot of

$$\frac{\Phi}{\Phi_1} = \frac{3}{2} - \frac{1}{R/R_G} - \frac{(\varsigma/R_G)^2}{2} + \frac{(z/R_G)^2}{6}$$

$$\Phi_1 = GM/R_G$$

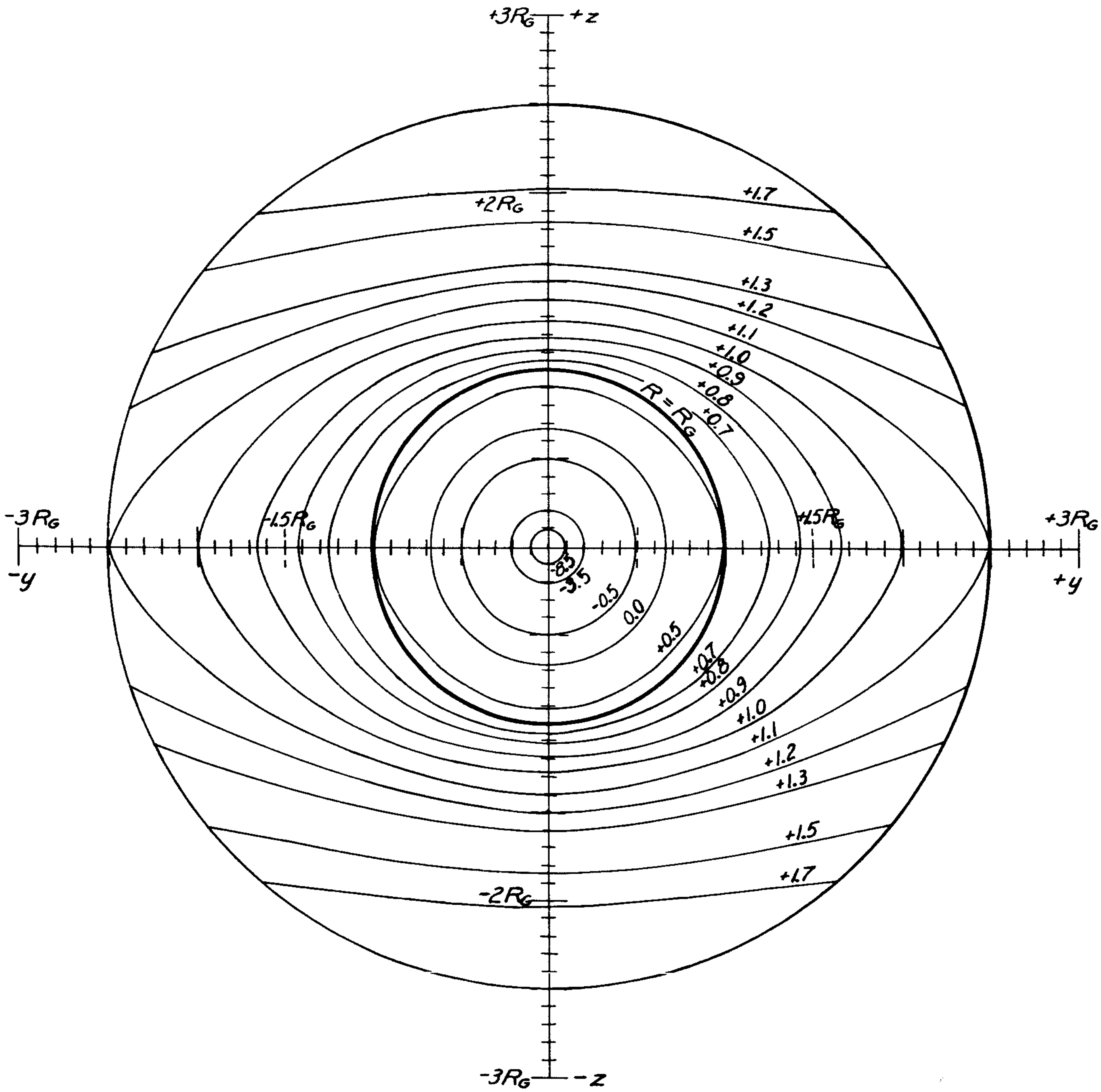
$$R_G = a\sqrt[3]{M/3M_c}$$

$R = R_G$ = Nominal Radius of the Gravisphere

Locus of Inflection Points of Constant Potential Lines

FIGURE 4

FIELD OF GRAVITATIONAL POTENTIAL
TANGENTIAL CYLINDER



Plot of

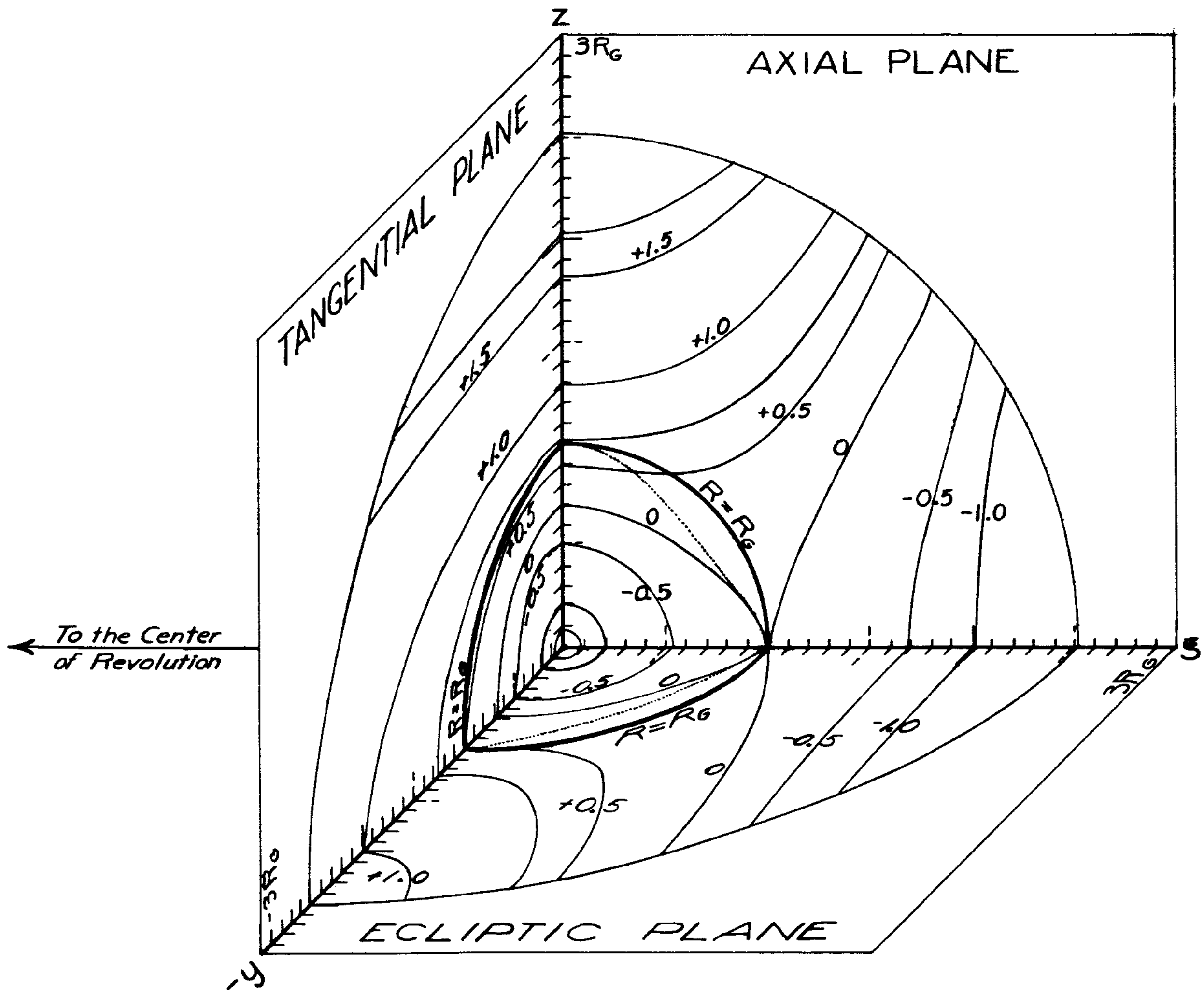
$$\frac{\Phi}{\Phi_1} = \frac{3}{2} - \frac{1}{R/R_G} + \frac{(z/R_G)^2}{6}$$

$$\Phi_1 = GM/R_G$$

$$R_G = a\sqrt[3]{M/3M_c}$$

FIGURE 5

FIELD OF GRAVITATIONAL POTENTIAL
ISOMETRIC VIEW



Plot of

$$\frac{\Phi}{\Phi_1} = \frac{3}{2} - \frac{1}{R/R_G} - \frac{(\xi/R_G)^2}{2} + \frac{(z/R_G)^2}{6}$$

$$\Phi_1 = GM/R_G$$

$$R_G = a\sqrt[3]{M/3M_c}$$

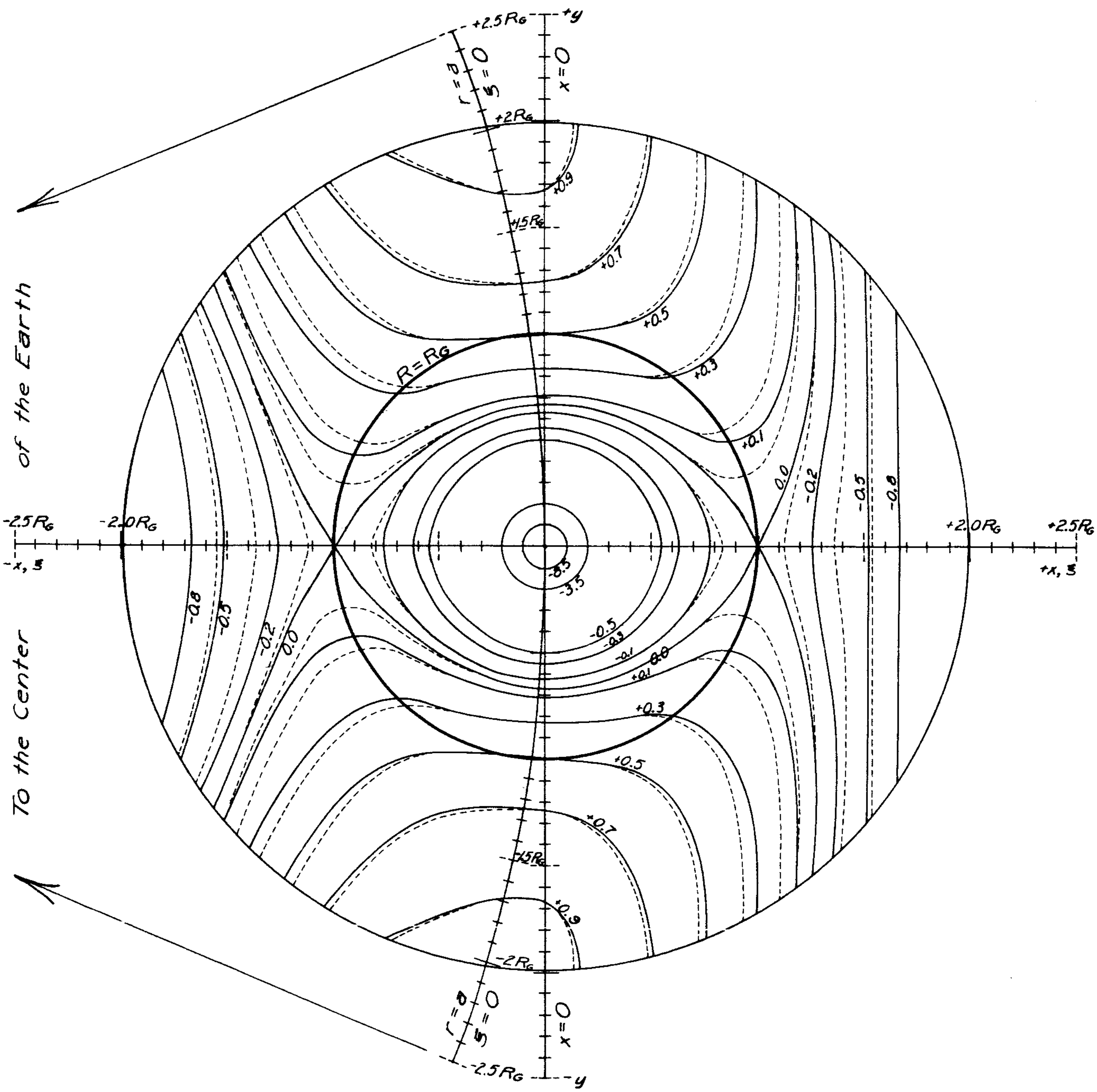
R = R_G = Nominal Radius of the Gravisphere (solid line)

Locus of Inflection Points of Constant Potential Lines (dotted line)

FIGURE 6

FIELD OF THE MOON'S GRAVITATIONAL POTENTIAL

ECLIPTIC PLANE



Plot of

$$\frac{\Phi}{\Phi_1} = \frac{3}{2} - \frac{1}{R/R_G} - \frac{(\xi/R_G)^2}{2} + \boxed{K\frac{(\xi/R_G)^3}{3}}$$

$$\Phi_1 = GM/R_G$$

$$K = R_G/a = \sqrt[3]{M/3M_E} = 1/6.25$$

FIGURE 7

7. Extent of Gravispheres

Figures 3 through 7 are intended to portray the way in which the gravitational fields of a large central body and a smaller central body interact. They are meant especially to illustrate the direction and magnitude of the gravitational accelerations to which infinitesimal masses m would be subjected in their interacting region. The directions of such accelerations are everywhere normal to the constant potential surfaces from higher to lower algebraic values of Φ (or from lower to higher negative values of Φ). In addition, their magnitudes are inversely proportional to the separation between the constant potential surfaces. The directions and spacings of the lines of constant values of Φ in these figures thus indicate the direction and magnitude of the components of the gravitational accelerations in each of their respective planes. In the case of the ecliptic planes represented in Figures 3 and 7, no other components (in the z direction) of the acceleration vector are involved.

Figures 3 through 7 are intended especially to provide a qualitative concept of the extent to which the gravitational attraction of a smaller revolving body M can be considered to be dominant. Toward this end, they clearly indicate that the neutral points can well be considered to be the limits of such dominance along the line which passes through the two bodies of interest. For example, small masses originally at rest with $v = 0$ on that line and in the absence of other forces, $\overleftarrow{F}/m$, would be subjected to an acceleration $d\overleftarrow{v}/dt$ of Equation 11 which is: zero at the neutral points; toward the smaller body M at all points inside the neutral points; and away from the smaller body M at all points outside the neutral points.

Similarly for other directions from the smaller body M, the combined gravitational acceleration evidently must be directed more toward than away from M if its attraction is to be considered to be dominant. As indicated in Figures 3 through 7, this particular criterion is satisfied throughout a very large region. As indicated in Figure 5, for example, it is satisfied throughout virtually the entire tangential plane or cylinder (for which $x = 0$ or $\rho = a$).

Clearly, then, additional criteria must be used to define the extent of gravispheres. For example, the Newtonian attraction of the Sun and the centrifugal force of the Earth-Sun system balance each other everywhere on the circle of radius r = A around the Sun. Consequently the net acceleration is theoretically directed directly toward the center of the Earth, irrespective of how feeble the Earth's attractive force, $-(GM/R^2)\hat{R}$, might be anywhere on this circle around the Sun.

To find a more practical definition, it was noticed while drawing these figures that the constant potential lines of Figures 3 and 4 tend to be parallel to the x-axis in the vicinity of a sphere of radius R_G around the smaller body. It was suggested thereby that the extremities of a revolving body's gravisphere might well be defined to be at those positions at which the component of the combined field of gravity in the direction of the central body vanishes. In other words, the smaller body's influence can be considered to be dominant wherever its presence results in a reversal of the sense of the component of acceleration $(\overleftarrow{f}\cdot\hat{r})$ in the direction $\hat{r}$ from that which it would have had in the absence of the smaller body. As indicated in Equation 22, that undisturbed sense (for 1/R = 0) is away from the $\xi = 0$ or r = a sphere. A symbolic expression of this criterion is thus that

$$(\overleftarrow{f}\cdot\hat{r})/\xi \leq 0 \tag{27}$$

for all points within or at the extremity of the small body's gravisphere.

A convenient approximate evaluation of this criterion can be obtained from Equation 22 by neglecting the higher order K terms, by considering the vector z to be at right angles to the vector r, and by noting that the cosine of the angle between the vectors R and r is very nearly equal to ξ/R. In that case, substituting $\overleftarrow{f}$ of Equation 22 in Equation 27 leads to the approximate criterion that

$$\left(- \frac{R_G^{\,2}}{R^2} \; \frac{\xi}{R} + \frac{\xi}{R_G} \right) \frac{1}{\xi} \leq 0$$

or that

$$R \leq R_G$$

The location of the limiting sphere of radius $R = R_G$ for this criterion is indicated by the heavy circles in Figures 3 through 7.

In addition, the mass of the smaller body can be considered to be "completely dominant" only where the field of gravitational attraction converges toward it, or only where the constant potential surfaces are concave toward it. As indicated in Figure 5, this criterion is evidently satisfied throughout the tangential plane. For any plane which passes through the x-axis with a dihedral angle α from the ecliptic plane as defined in Figure 2, the limit of this convergence occurs at those inflection points where the slopes, $(d\eta/dx)_\Phi$, of constant potential lines do not change over an increment distance ds along them, or where

$$\frac{d}{ds}\,(d\eta/dx)_\Phi = 0 \tag{28}$$

and η is defined by the equations

$$\eta^2 = y^2 + z^2; \qquad \sin\alpha = z/\eta \quad , \quad R^2 = x^2 + \eta^2 \tag{29}$$

Since the change of Φ in any direction can be expressed as

$$d\Phi = (\partial\Phi/\partial x)dx + (\partial\Phi/\partial y)dy$$

the slope of the direction of constant Φ lines, along which $d\Phi = 0$, are given by

$$(d\eta/dx)_\Phi = -\,\frac{\partial\Phi/\partial x}{\partial\Phi/\partial\eta} \tag{30}$$

Consequently Equation (28) can be expanded as

$$\frac{\partial}{\partial x}\left(-\,\frac{\partial\Phi/\partial x}{\partial\Phi/\partial\eta}\right)\frac{dx}{ds} + \frac{\partial}{\partial\eta}\left(-\,\frac{\partial\Phi/\partial x}{\partial\Phi/\partial\eta}\right)\frac{d\eta}{ds} = 0$$

or, by carrying out the differentiations and clearing fractions, the locus of inflection points occurs where

$$\frac{\partial^2\Phi}{\partial x^2}\left(\frac{\partial\Phi}{\partial\eta}\right)^2 - 2\,\frac{\partial^2\Phi}{\partial x\partial\eta}\left(\frac{\partial\Phi}{\partial x}\right)\left(\frac{\partial\Phi}{\partial\eta}\right) + \frac{\partial^2\Phi}{\partial\eta^2}\left(\frac{\partial\Phi}{\partial x}\right)^2 = 0 \tag{31}$$

A convenient approximation of the corresponding locus of inflection points can then be obtained by neglecting the higher order K terms and neglecting differences between values of ξ and x in Equation 25, or by using the approximate formula that

$$\frac{\Phi - \Phi_o}{GM/R_G} \doteq -\frac{R_G}{R} - \frac{1}{2}\frac{x^2}{R_G{}^2} + \frac{1}{6}\frac{\eta^2 \sin^2\alpha}{R_G{}^2}$$

to obtain the expressions

$$\frac{\partial\Phi}{\partial x} = \frac{xR_G}{R^3} - \frac{x}{R_G{}^2} \; ; \qquad \frac{\partial^2\Phi}{\partial x^2} = -\frac{3x^2 R_G}{R^5} + \frac{R_G}{R^3} - \frac{1}{R_G{}^3} \; ;$$

$$\frac{\partial\Phi}{\partial\eta} = \frac{\eta R_G}{R^3} + \frac{\eta\sin^2\alpha}{3R_G{}^2} \; ; \qquad \frac{\partial^2\Phi}{\partial\eta^2} = -\frac{3\eta^2 R_G}{R^5} + \frac{R_G}{R^3} + \frac{\sin^2\alpha}{3R_G{}^2} \; ;$$

$$\frac{\partial^2\Phi}{\partial x\partial\eta} = -\frac{3x\eta R_G}{R^5}$$

for substitution in Equation 31. The result can be arranged into the expression

$$\frac{R^3}{R_G{}^3} \leq 1 - \frac{3\cos^2\theta\sin^2\theta[1 + (\sin^2\alpha)/3]^2}{\left(\frac{R_G{}^3}{R^3} + \frac{\sin^2\alpha}{3}\right)\left(\frac{R_G{}^3}{R^3} + \frac{\sin^2\alpha}{3}\sin^2\theta - \cos^2\theta\right)} \tag{32}$$

for those radii, R, at which the accelerations are convergent toward the smaller body at various angles, θ, from the x-axis for which, as shown in Figure 2, $\cos\theta = x/R$ and $\sin\theta = \eta/R$. The limiting extent of such convergence is indicated by the dotted lines in Figures 3, 4 and 6 and by the heavy circle in Figure 5 of the tangential plane in which $\cos\theta = 0$ and hence $R \leq R_G$.

8. Conclusions

In summary, those regions of the universe in which orbital operations can be usefully considered as being Earth-centered, Moon-centered or Mars-centered could well be called the outermost "pyrospheric" portions of the "gravispheres" of the Earth, Moon, or Mars, etc. Quantitatively, the extent of the gravisphere of any such celestial body could apparently be well defined as being that region in which the gravitational accelerations of the infinitesimal masses of artificial satellites are dominated by that body's mass in the sense that:

(1) The components of acceleration along that body's radius vector are directed toward (rather than away from) it;

(2) The components of acceleration along the radius vector of the larger central mass around which it revolves are in a direction opposite to that which would exist in the absence of the body of interest; and

(3) The two-body field of gravitational accelerations converges (or the two-body field of gravitational potential is concave) toward the body of interest.

Much of the probable utility of this concept of "gravispheres" derives from the fact that their extents are closely approximated by spheres of radius equal to the "characteristic gravispheric radius, R_G" defined by the equation

$$R_G = a\sqrt[3]{M/3M_c} \qquad (14)$$

That is, for most purposes gravispheres can readily and usefully be thought of as a sphere of that radius. In case more accurate evaluations of the second criterion are desired, the value of R_G can well serve as a first approximation as illustrated with Equations 15 and 16. In those cases in which the third or convergence criterion might be important, the nearly spherical shape of the "nominal gravispheres" can readily be visualized as having been flattened by about 15 percent as indicated by the dotted lines in Figures 3, 4 and 6.

The concept of gravispheres and of their characteristic radii, R_G, seem likely to prove to be especially useful in terms of the convenient normalized form of the equations of motion with which they are associated. That is, the "exact" equations of motion in the form

$$\frac{\overleftarrow{f}}{GM/R_G^2} = \frac{d\overleftarrow{v}/dt + 2\overleftarrow{\omega} \times \overleftarrow{v} - \overleftarrow{F}/m}{GM/R_G^2}$$

$$= \frac{-\overleftrightarrow{R}}{(R/R_G)^2} + \frac{\xi}{R_G}\overleftrightarrow{r} - \frac{z}{3R_G}\overleftrightarrow{z}$$

$$- K\left(\frac{\xi}{R_G}\right)^2 \frac{(1 + 2K\xi/3R_G)}{(1 + K\xi/R_G)^2}\overleftrightarrow{r} + K^2[\overleftrightarrow{r} - \overleftrightarrow{x}] + K^3\left[\frac{\xi}{R_G}\overleftrightarrow{r} - \frac{z}{R_G}\overleftrightarrow{z}\right]$$

$$= - \frac{1}{R_G}\overleftarrow{\nabla} \frac{\Phi}{GM/R_G} \qquad (33)$$

and

$$\frac{\Phi-\Phi_o}{GM/R_G} = - \frac{1}{R/R_G} - \frac{(\xi/R_G)^2}{2} + \frac{(z/R_G)^2}{6}$$

$$+ \frac{K}{3}\frac{(\xi/R_G)^3}{1 + K\xi/R_G} - K^2\left[\frac{\xi}{R_G} - \frac{x}{R_G}\right] - \frac{K^3}{2}\left[\left(\frac{\xi}{R_G}\right)^2 - \left(\frac{z}{R_G}\right)^2\right] \qquad (25)$$

provide for especially convenient numerical computations since the "characteristic gravispheric constant, $K = R_G/a = \sqrt[3]{M/3M_c}$ ", is usually much smaller than unity. Hence for many purposes all terms involving K can be neglected and the inclusion of whatever higher order terms in K might be needed for other purposes involves only small fractional corrections of first order approximations obtained by setting $K = 0$.

The most recent values of astronomical constants listed in Table I have been used to compute the values of $K = R_G/a$, $1/K = a/R_G$, and the size of R_G with respect to the "body-radii, R_o", the astronomical unit distance, A, and meters, m, for each of the planets and the Moon listed in Table II.

In addition, the corresponding characteristic units of acceleration, or force per unit mass, and of potential have been computed and listed in Table III. These tables of numerical values are meant to lend substance to the concept of gravispheres as well as to provide for convenient numerical use of Equations 33, 25 and 20.

The quantitative visualization and utilization of these concepts and equations are especially convenient for the Earth-Sun and Moon-Earth pairs of bodies. As illustrated in Figure 8 and listed in Table IV, the geometric relationships among their body sizes, gravispheric radii and revolving distances can be approximated exceptionally well with small, easily remembered and used, few-digit numbers. That is, the Earth's nominal gravispheric radius, R_{Ge}, is almost exactly equal to 235 times a typical radius of the Earth and almost exactly 1/100 of the distance from the Earth to the Sun. Similarly, the mean revolving distance of the Moon is almost exactly 60 3/8 times a typical radius of the Earth and 6 1/4 times the Moon's gravispheric radius, R_{Gm}, which in turn is very nearly (60 3/8)/(6 1/4) or 9.66 times a typical radius of the Earth. The radius of the Moon is very nearly the easily remembered fraction, 0.273, of a typical radius of the Earth. Finally, the characteristic accelerations GM_e/R_{Ge}^2, GM_m/R_{Gm}^2 and GM_e/a_m^2 involved in Equations 33, 25 and 20 are very closely approximated by the values of 0.178, 1.3 and 2.7 millimeters per second per second, respectively.

Such characteristic dimensions of the Earth's and Moon's orbits and gravispheres can be utilized to fullest advantage by using them to define corresponding units of distance in much the same way that the length of the meter was defined so that it is typical of a 1/10,000,000 part of a quadrant of the Earth. It is thus suggested here that units of distance be defined as in Table IV for, especially, convenient routine use in orbital operations. With such units of distance and acceleration, for example, the denominators of the normalized length and acceleration factors of Equations 33, 25 and 20 could be considered to be unity for all but the most precise of purposes. The small fractional corrections required to correct such unity approximations for those precise purposes are listed in Table V.

GEOSPHERIC AND LUNASPHERIC UNITS OF DISTANCE

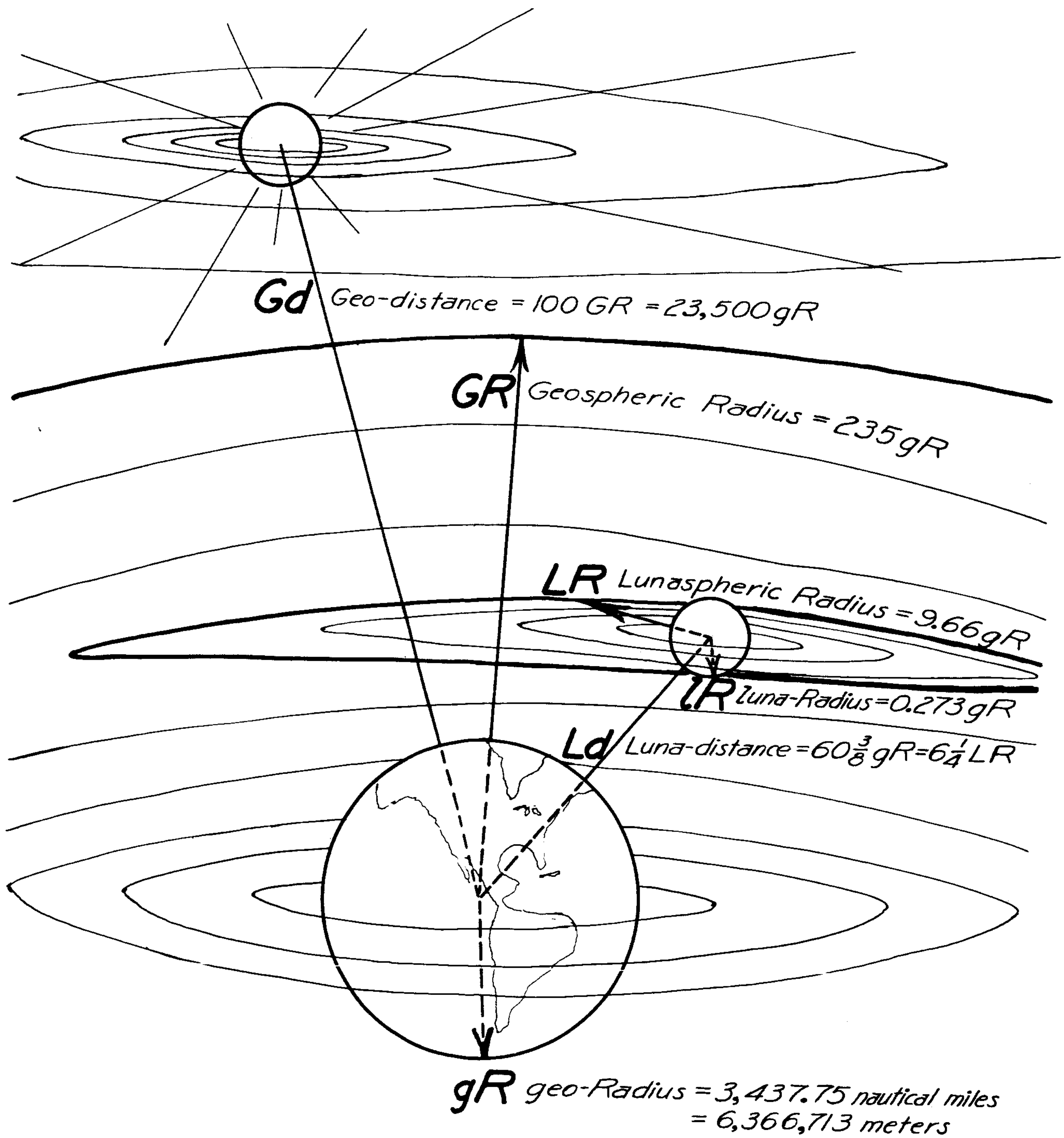


FIGURE 8

Briefly, these definitions are based upon the definition of the International Nautical Mile as being 1,852 meters (or, more fundamentally, as being (1,852)(1,650,763.73) wave-lengths of the primary standard Kr 86 measure of length) and on the recently proposed[4] "nautical rad" of exactly 3,437.75 (or very nearly the 21,600 minutes of arc per circumference divided by 2π) nautical miles. It is expected that the "characteristic gees" of 0.178, 1.3 and 1.7 mm/s^2 will be found to serve the same kind of useful purpose for gravispheric operations that the "standard gravity, $g_o = 9.80665\ m/s^2$" now serves operations on the surface of the Earth.

The symbols and names of the units listed in Table IV and used in Table V are suggested here only tentatively; the selection of appropriate symbols and names for such units has been found to be the most difficult part of the definition of such new units. For example, names such as "geosec, geomin, geodeg, geoquad and geocirc" now seem preferable to the recently suggested[4] "nautical" units with corresponding suffixes. Especially, it now seems better to identify the typical radius of the Earth with the name "georadius" and the symbol, gR, than with the name "nautical rad". In this regard, the convention has been tentatively introduced here to use small letter prefixes such as "g" and "l" to denote the visible, solid or lithospheric, aspects of the "geo" Earth and the "luna" Moon, and to use corresponding capital letters such as "G" and "L" to denote the all-inclusive gravispheric (or Geospheric and Lunaspheric) aspects of the operational regions associated with those bodies.

Finally, it is important to recognize the fact that the astronomical constants evaluated in Table V are themselves more "standard average values" than the value of any specific distance or acceleration which might exist and be of interest at any specific time for some specific purpose. The sizes of the suggested units of length and acceleration thus seem likely to be as typically representative of any such specific value as any other one standard value might be.

It is thus expected and hoped that these simple defining numbers will be found to be conveniently useful for both

(1) routine orbital operations in which differences between these typical unit values and individual values will frequently be negligible, and

(2) scientific determinations of more precise average and specific values of related astronomical constants based on non-redundant computations in terms of small-difference corrections of these simply defined and conceived, or "ideal", unit values.

TABLE I

ASTRONOMICAL CONSTANTS

Sources: 1. "Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac" Nautical Almanac Offices of the United Kingdom and the United States of America, London, 1961

2. "Report to the Executive Committee of the Working Group on the System of Astronomical Constants" International Astronomical Union, 28 February 1964

Revolving Body	$\frac{M}{M_c} = \frac{\text{Mass of Sun (or Earth)}}{\text{Mass of Planet (or Moon)}}$	$\frac{a}{A} = \frac{\text{Mean Revolving Distance}}{\text{Astronomical Unit}}$	SD = Equatorial Semi-Diameter, Seconds of Arc at Unit Distance, A.	Gravitational Constant, GM: $10^{12} m^3/s^2$
Mercury	6,000,000	0.387 099	3.34	22.
Venus	408,000	0.723 332	8.41	325.
Earth	332,958	1.000 000	8.794 05	398.603
Mars	3,093,500	1.523 691	4.68	42.902
Jupiter	1,047.355	5.202 803	98.47	126,717.3
Saturn	3,501.6	9.538 843	83.33	37,902
Uranus	22,869	19.181 951	34.28	5,803.4
Neptune	19,314	30.057 779	36.56	6,871.6
Pluto	360,000	39.438 71	10 ?	370.
Moon	81.30	0.002 569 52	2.396 3	4.903

Astronomical Unit of Distance, A = 149,600 x 10^6 meters

Equatorial Radius of Earth, R_{oe} = 6,378,160 meters

Perturbed Mean Distance of the Moon = 384,400 x 10^3 meters

Semi-Diameter of the Moon at Mean Distance = 15'32.6"

TABLE II

PROPORTIONS OF GRAVISPHERES

Revolving Body	$K = R_G/a = \sqrt[3]{M/3M_c} = \frac{\text{(Nominal) Radius of Gravisphere}}{\text{Mean Revolving Distance}}$	$1/K = a/R_G = \sqrt[3]{3M_c/M} = \frac{\text{Mean Revolving Distance}}{\text{(Nominal) Radius of Gravisphere}}$	$R_G/R_o = \frac{\text{(Nominal) Radius of Gravisphere}}{\text{Equatorial Radius of (Solid) Body}}$	$R_G/A = \frac{\text{(Nominal) Radius of Gravisphere}}{\text{Astronomical Unit Distance}}$	$R_G/10^8 m = \frac{\text{(Nominal) Radius of Gravisphere}}{\text{100,000,000 meters}}$
Mercury	0.003 8	260	91	0.001 5	2.2
Venus	0.009 35	107.0	165.8	0.006 76	10.12
Earth	0.010 003 73	99.962 5	234.638 5	0.010 003 73	14.965 62
Mars	0.004 758 55	210.147	320	0.007 250 6	10.846 9
Jupiter	0.068 274 98	14.646 65	744.08	0.355 221 3	531.411 0
Saturn	0.045 660 1	21.900 9	1,078.1	0.435 545	651.57
Uranus	0.024 427 4	40.938	2,819	0.468 56	700.97
Neptune	0.025 842 6	38.696	4,382	0.776 77	1,162.05
Pluto	0.009 7	102	8,000 ?	0.38	575
Moon	0.160 05	6.247 9	35.399	0.000 411 26	0.615 24

TABLE III

GRAVITATIONAL CONSTANTS

Revolving Body	Potential GM_c/a $10^6 m^2/s^2$	Potential GM/R_G $10^3 m^2/s^2$	Acceleration GM_c/a^2 $10^{-6} m/s^2$	Acceleration $GM/R_G{}^2$ $10^{-6} m/s^2$	Acceleration $GM/R_o{}^2$ m/s^2
Mercury	2,291.8	100	39,575.1	450	3.8
Venus	1,226.48	322	11,334.2	318	8.74
Earth	887.152	266.346	5,930.16	177.972	9.798 3
Mars	582.239	39.553	2,554.30	36.465	3.724
Jupiter	170.514 4	2,384.55	219.075	44.872 0	24.844
Saturn	96.939 9	581.70	65.174 2	8.927 6	10.376
Uranus	46.249 3	82.791	16.116 9	1.181 1	9.388
Neptune	29.514 9	59.134	6.563 76	0.508 87	9.773
Pluto	22.494 5	6.4	3.812 60	0.111	7 ?
Moon	1.036 95	79.69	2,697.58	1,295.3	1.623 1

TABLE IV

SUGGESTED
GEOSPHERIC AND LUNASPHERIC UNITS OF DISTANCE AND ACCELERATION

gR or "georadius" = 3,437.75 "geomins" or nautical miles, gM,
corresponding to 21,000/2π nautical miles
= 3,437.75 (1,852) or 6,366,713 meters, m.

GR or "geospheric radius" = 235 georadii, gR
= 235 (6,366,713) or 1,496,177,555 meters, m.

Gd or "geodistance" = 100 geospheric radii, GR
= 23,500 georadii, gR
= 149,617,755,500 meters, m.

lR or "lunaradius" = 0.273 georadii, gR
= 0.273 (6,366,713) or 1,738,122.649 meters, m.

LR or "lunaspheric radius" = 9.66 georadii, gR
= 9.66 (6,366,713) or 61,502,447.58 meters, m.
= 9.66/0.273 or 46/1.3 or 35.384 61··lunaradii, lR.

Ld or "lunadistance" = 60.375 or 60 3/8 georadii, gR
= 60.375 (6,366,713) or 384,390,297.375 meters, m
= 6.25 lunaspheric radii, LR
= 575/2.6 or 221.153 8··lunaradii, lR

Gg or "geospheric gee" = 0.000 178 meters/(ephemeris second)^2, m/es^2
= 0.000 888 105··(geospheric radii)/(ephemeris day)^2, GR/ed^2
$\sim$ Keplerian period, $2\pi/\sqrt{Gg/GR}$, of 211.319 ed

Lg or "lunaspheric gee" = 0.001 3 meters/(ephemeris second)^2, m/es^2
= 0.157 790··(lunaspheric radii)/(ephemeris day)^2, LR/ed^2
$\sim$ Keplerian period, $2\pi/\sqrt{Lg/GR}$, of 15.817 2 ed

Ldg or "lunar distance gee" = 0.002 7 meters/(ephemeris second)^2, m/es^2
= 0.052 434 7··(lunar distances)/(ephemeris day)^2, Ld/ed^2
$\sim$ Keplerian period, $2\pi/\sqrt{Ldg/Ld}$, of 27.438 5··ed.

TABLE V

GEOSPHERIC AND LUNASPHERIC VALUES OF ASTRONOMICAL CONSTANTS

Source: "Report to the Executive Committee of the Working Group on the System of Astronomical Constants" International Astronomical Union, 28 February 1964

Equatorial Radius of the Earth, $R_{oe} = 6{,}378{,}160(1 \pm 0.000\ 013)\text{m}$
$= 1(1 + 0.001\ 883 \pm 0.000\ 013)\text{gR}$

Gravispheric Radius of the Earth, $R_{Ge} = 1.496\ 562(1 \pm 0.000\ 021)10^9\text{m}$
$= 1(1 + 0.000\ 256 \pm 0.000\ 021)\text{GR}$

Astronomical Unit, $A = 1.496\ 00(1 \pm 0.000\ 013)10^{11}\text{m}$
$= 1(1 - 0.000\ 118 \pm 0.000\ 013)\text{Gd}$

Radius of the Moon, $R_{om} = 1.738\ 02(1 \pm 0.000\ 1)10^6\text{m}$
$= 1(1 - 0.000\ 05 \pm 0.000\ 10)\text{lR}$

Gravispheric Radius of the Moon, $R_{Gm} = 6.152\ 42(1 \pm 0.000\ 04)10^7\text{m}$
$= 1(1 + 0.000\ 35 \pm 0.000\ 04)\text{LR}$

Perturbed Mean Distance of Moon, $a_m = 384{,}400(1 \pm 0.000\ 003)\text{m}$
$= 1(1 + 0.000\ 026 \pm 0.000\ 003)\text{Ld}$

Geospheric Constant of Acceleration, $GE/R_{Ge}^2 = 1.779\ 718(1 \pm 0.000\ 043)10^{-4}\text{m/s}^2$
$= 1(1 - 0.000\ 164 \pm 0.000\ 043)\text{Gg}$

Lunaspheric Constant of Acceleration, $GM_m/R_{Gm}^2 = 1.295\ 263(1 \pm 0.000\ 042)10^{-3}\text{m/s}^2$
$= 1(1 - 0.003\ 644 \pm 0.000\ 042)\text{Lg}$

Earth Acceleration at Lunar Distance, $GE/a_m^2 = 2.697\ 576(1 \pm 0.000\ 009)10^{-3}\text{m/s}^2$
$= 1(1 - 0.000\ 898 \pm 0.000\ 009)\text{Ldg}$